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DISCRETIZATION OF INCOMPRESSIBLE VORTICITY–VELOCITY EQUATIONS ON TRIANGULAR MESHES

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SUMMARY

This paper describes a new approach to discretizing first- and second-order partial differential equations. It combines the advantages of finite elements and finite differences in having both unstructured (triangular/tetrahedral) meshes and low-order physically intuitive schemes. In this 'co-volume' framework, the discretized gradient, divergence, curl, (scalar) Laplacian, and vector Laplacian operators satisfy relationships found in standard vector field theory, such as a Helmholtz decomposition. This article focuses on the vorticity–velocity formulation for planar incompressible flows. The algorithm is described and some supporting numerical evidence is provided.

KEY WORDS Navier–Stokes equations Vorticity–velocity formulation Voronoi diagram Complementary volumes

1. INTRODUCTION

Many flow problems need triangular/tetrahedral meshes for their solution. Included are problems with complex geometry and problems requiring fully adaptive meshes. Finite elements occur naturally in this context. Higher-order elements are frequently used but are quite complex in terms of implementation and can hide the underlying physics. The main difficulties associated with using low-order elements concern the existence of visible and invisible spurious modes.[1,2] Not all low-order elements suffer from spurious mode problems, but those which do not can compromise efficiency. To illustrate, consider a uniform square mesh. Instead of the unstable bilinear–constant element pair we can use a pressure field which is piecewise constant on non-overlapping 2×2 blocks of mesh cells. This element pair is stable[2] but may provide inadequate resolution of the pressure. Or consider penalty schemes. These make the numerical solution depend on a parameter whose precise effect on the solution is often hard to assess. A third approach uses a pressure field (e.g. continuous piecewise linear) with a higher degree of smoothness than is required by the standard weak formulation of the Navier–Stokes equations. Nevertheless, the computed pressure is usually only first-order accurate instead of second-order which one might expect from approximation theory. Thus the potential extra accuracy is sacrificed for stability. These observations are not intended to belittle any particular approach. Rather, they indicate opportunities for new ideas and approaches. Incidentally, low-order triangular schemes for compressible flow are already common and several schemes are widely

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used.[3,4] These schemes are designed for conservation laws and do not naturally generalize to incompressible flow.

In this paper we present an approach based on *complementary volumes*. The goal is to produce stable and physically intuitive discretizations for incompressible flows. The main ideas were presented in Reference 5 for the primitive variable equations. The emphasis here is on the vorticity–velocity formulation for planar problems. Alternative vorticity–velocity techniques can be found e.g. in References 6 and 7.

Basic to the technique is the use of dual pairs of mesh systems. The dual meshes are related by a kind of orthogonality. An extra restriction must be imposed on the mesh pairs to make them convenient for use. These mesh issues are the subject of the next section. Section 3 covers discretization of the div–curl system familiar in the vorticity–velocity context. In addition to the method of discretization, we summarize the content of some rigorous error estimates which have been proved recently. In Section 4 a time discretization suitable for the vorticity transport equation is given. Section 5 discusses the all-important boundary conditions for the vorticity. One attractive feature of the complementary volume approach is a close parallel with standard vector field theory. For illustration we will derive the vorticity–velocity equations from the primitive variable equations by reductions similar to the continuous ones. This is given in Sections 6 and 7.

2. COMPLEMENTARY VOLUMES

We will deal with dual sets of control volumes. In three dimensions these are required to have the property that the edges of both sets of control volumes are perpendicular to the faces of the other set of control volumes. Such pairs of control volumes will be called *complementary volumes* or *co-volumes* for short. We will also apply this terminology in two dimensions. The simplest example consists of two sets of squares of the same size, where the corners of one of the sets are positioned at the centres of the other and the relation is mutual. More interesting is the situation depicted in Figure 1. Here one set of control volumes consists of triangles and to construct the other (dual) tesselation we join the circumcentres of the triangles surrounding each node of the triangulation. In the figure, C_1–C_6 are the circumcentres and the polygon with these vertices is the control volume. C_5 is outside its triangle because the triangle is obtuse. The orthogonality property

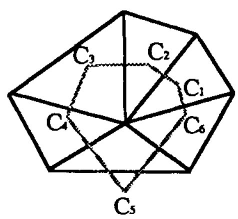

Figure 1

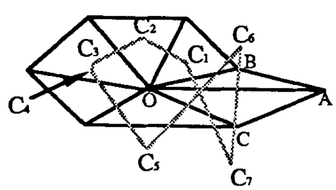

Figure 2

follows from elementary geometry. In this case the triangle edges are bisected by the polygon edges. We remark that in three dimensions the circumcentres of tetrahedra are joined across adjacent faces and the joining lines from the edges of polyhedra around the nodes of the triangulation (tesselation). These polyhedra form the dual control volumes. The tetrahedral edges are perpendicular to the polyhedral faces. It is a fact that the polyhedral edges are perpendicular to the tetrahedral faces and pass through the circumcentres of these faces.

The above *normal dual* tesselations do not always provide a good set of control volumes for a triangulation. The main difficulty is illustrated in Figure 2, where the control volume C_1–C_7 is self-intersecting. The existence of such control volumes complicates the application of the discretizations of Section 3 and we shall not consider them in this paper. Specifically, we will restrict attention to a class of triangulations called *locally equiangular*. A triangular mesh is called locally equiangular (LE)[8] if the sum of the angles opposite the diagonal of every convex quadrilateral made from two adjacent triangles is less than or equal to 180°. For example, every acute triangulation is LE. Also, choosing BC as an edge of the mesh instead of OA in Figure 2 makes this mesh LE. A straightforward geometrical argument shows that if a mesh is LE, then its interior co-volumes are convex, non-overlapping and cover the flow domain Ω except possibly for a strip near its boundary Γ. Hence they are suitable control volumes for the interior of the domain.

The resulting triangulation and its dual are very close to a Delaunay–Voronoi mesh pair. If we were considering the whole plane R^2 and not merely the subset Ω there would be no question about this. The presence of the boundary Γ calls for further discussion. Basically, the classical definition of a Delaunay triangulation as the dual of the Voronoi diagram of a set of points ensures that the former is a triangulation of the *convex hull* of the points. This does not necessarily coincide with Ω. None of this matters in this paper. The only requirement is that the mesh should be LE. The required properties of the co-volumes can be deduced without any further assumptions, and Ω is not required to be convex.

We will denote edge lengths of the triangulation by h_i and those of the normal dual by h'_i. For some purposes, it is convenient to add extra nodes to the normal dual. These are placed at the midpoints (and circumcentres) of the boundary edges of the triangulation. The dual edge associated with a boundary edge extends from the boundary dual node to the circumcentre of its triangle. The length of this edge is still denoted by h'_i.

3. DISCRETE OPERATORS

In this section we will use the complementary mesh system to discretize the 'div,' 'curl,' 'grad' and 'scalar Laplacian' operators. Referring to the triangle T in Figure 3, let u_1, u_2, u_3 be velocity components at the midpoints of the edges directed along the unit normals $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ respectively. We approximate

$$\int_T \operatorname{div} \mathbf{u} \, dT = \int_{\partial T} \mathbf{u} \cdot \mathbf{n} \, ds \approx u_1 h_1 + u_2 h_2 + u_3 h_3. \quad (1)$$

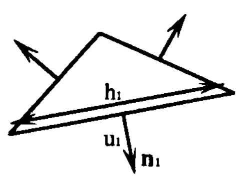

Figure 3

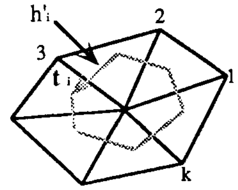

Figure 4

Now refer to the co-volume T' in Figure 4. Let u_i denote the velocity components at the triangle midpoints along the unit tangents $\mathbf{t}_i$, $i = 1, \ldots, k$. We approximate

$$\int_{T'} \operatorname{curl} \mathbf{u}\, \mathrm{d}T' = \int_{T'} (\partial_x v - \partial_y u)\, \mathrm{d}T' = \int_{\partial T'} \mathbf{u} \cdot \mathbf{t}\, \mathrm{d}s \approx \sum_{i=1}^{k} u_i h_i'. \tag{2}$$

In general, there may be any number of sides $\geqslant 3$ to the co-volume.

A basic property of the flux approximation (1) and the circulation approximation (2) is that they use the *same* velocity components u_i (properly signed). Both approximations are made using only the normal velocity components relative to the triangles.

These discretizations can be used to approximate the typical velocity calculation

$$\operatorname{div} \mathbf{u} = 0,$$

$$\operatorname{curl} \mathbf{u} = \omega,$$

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

where for brevity we assume Ω is simply connected and ω is the prescribed vorticity field.

Apply (1) in each triangle of the mesh to get the equations

$$Fu = 0. \tag{3}$$

Apply (2) to each interior co-volume to get the equations

$$Cu = \tilde{\omega}, \tag{4}$$

where a typical component of $\tilde{\omega}$ is

$$\tilde{\omega}_i = \int_{T'} \omega\, \mathrm{d}T'.$$

The normal boundary components are set to zero. We will write this as

$$u|_{\partial\Omega} = 0. \tag{5}$$

It is proved in Reference 9 that (3)–(5) always has a unique solution, and an error estimate of the form

$$\| u - m(\mathbf{u}) \|_W \leqslant C \max(h, h') |\mathbf{u}|_1 \tag{6}$$

holds. In (6), h and h' are the maximum mesh lengths for the mesh and co-mesh, and $|\mathbf{u}|_1$ is the Sobolev semi-norm on $\vec{H}^1(\Omega)$. On the left side, $m(\mathbf{u})$ is the set of average normal velocity components on the triangle sides of the exact solution; u denotes the computed components. The norm $\| \cdot \|_W$ is essentially an $L^2(\Omega)$-norm on the mesh. Thus the approximate method is first-order accurate. Equation (6) remains true for multiply connected domains and inhomogeneous data.

Approximation of the divergence is achieved by normalizing the fluxes by the triangle areas. We will denote the normalized fluxes by $\hat{F}u$. Similiarly, the normalized circulations which approximate the curl will be denoted by $\hat{C}u$. The normalizing constants for the circulations are the areas of the co-volumes.

Next, consider the discrete gradient operators. In the co-volume approach we define only certain *components* of these vectors. First we consider scalar-valued mesh functions defined at the nodes of the triangulation and denoted by ψ_i. For these functions we can take a finite difference along an edge of the triangulation. A suitable sign convention for this is to define the positive direction on the edge from low to high node number. Denoting by h_{ij} the distance between connected nodes i and j, where $i < j$, the finite difference is $(\psi_j - \psi_i)/h_{ij}$. Using a matrix T, these differences can be expressed as $T\psi$. For a given ψ we choose to interpret $T\psi$ as a set of velocity components u_k pointing to the right when the corresponding edge is viewed along its positive direction. Note that this orients the co-edge. By convention, the direction on it defined in this way is the positive direction. It is easy to check that with these sign conventions we always have $F(T\psi) = 0$. Conversely, it can be proved[9] that if $Fu = 0$ then $u = T\psi$. In other words, *ψ is a discrete stream function for F, and T is the associated discrete curl operator.* We have here a discrete analogue of the vector identity 'div curl $\mathbf{u} = 0$'.

Analogous to ψ we can define scalar-valued functions ϕ at the nodes of the dual mesh, including the boundary nodes. Using the orientation in the previous paragraph, ϕ can be differenced along an edge as $(\phi_k - \phi_l)/h'_{kl}$. This time, let S denote the differencing matrix, and interpret $S\phi$ as a vector component along the dual edge. It is clear that $C(S\phi) = 0$. It can also be proved that if $Cu = 0$ then $u = S\phi$. In this case we can choose ϕ to be zero (at the nodes) on Γ. This time, ϕ is a discrete velocity potential for C, and S is the associated discrete gradient operator. The continuous analogy is 'curl grad $\phi = 0$'. These results extend readily to multiply connected domains.

Now we will see how the above discretizations show up in second-order problems. First, consider the Poisson problem

$$\Delta\omega = g \quad \text{in } \Omega,$$

$$\omega = 0 \quad \text{on } \Gamma.$$

Integration over a co-volume T' gives

$$\int_{T'} \Delta\omega \, \mathrm{d}T' = \int_{\partial T'} \frac{\partial \omega}{\partial n} \, \mathrm{d}s = \int_{T'} g \, \mathrm{d}x \, \mathrm{d}y.$$

Assuming that ω is defined at the nodes of the triangulation, approximate the line integral as

$$\int_{\partial T'} \frac{\partial \omega}{\partial n} \, \mathrm{d}s \approx \sum_{i=1}^{k} \frac{\omega_i - \omega_0}{h_i} h'_i.$$

The notation is that of Figure 4. This can be expressed in terms of the above operators as the product $CT\omega$. In the next section we will use this approximation and denote it by $L_s\omega_0$. The important point here is that the Laplacian is approximated in terms of co-volume operators (suitably scaled). In this sense we have gone from discretization of first-order to second-order derivatives. There is another scaled approximation to the Laplacian associated with the normal dual mesh. It does not appear again in this paper, so we will merely point out that it is obtained as the product FS. It is worth noting that these Laplacians correspond to discretizations of the factorings $-(-\partial_y, \partial_x)(\partial_y, -\partial_x)^{\mathrm{T}}$ and $-(\partial_x, \partial_y)(-\partial_x, -\partial_y)^{\mathrm{T}}$.

4. CONVECTIVE DIFFERENCING

Section 3 contains the spatial discretizations required for the vorticity–velocity equations. In this section we will discretize the convection term. This term can be regarded as containing spatial differences or as a (material) time derivative. It would be possible to use the first interpretation for differencing, but in this paper we will use the second. The advantage is that the artificial dissipation produced will be mainly limited to the streamwise direction. Consequently, this part of the discretization is independent of the results in Section 3.

The vorticity transport equation is

$$\omega_t + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\Delta\omega + \tilde{f}, \tag{7}$$

where $\tilde{f} := \operatorname{curl}\mathbf{f}$ and $\mathbf{f}$ denotes the body force in the primitive variable equations. In terms of the material derivative $\mathrm{d}/\mathrm{d}t$ we have

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \nu\Delta\omega + \tilde{f}.$$

Schemes for this sort of problem go back to Reference 10 ($\nu = 0$); see also Reference 11. Our approach is slightly different, being based on material volumes. Thus integrating (7) over a material volume $V(t)$ gives, since $\operatorname{div}\mathbf{u} = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \omega\,\mathrm{d}x\,\mathrm{d}y = \nu\int_{\partial V(t)} \frac{\partial\omega}{\partial n}\,\mathrm{d}s + \int_{V(t)} \tilde{f}\,\mathrm{d}x\,\mathrm{d}y. \tag{8}$$

Since it is the integral on the left which is needed for (4) and the velocity equations, it seems reasonable to discretize (8) directly.

For this, choose $V(t) = T'_i$, where T'_i denotes the co-volume containing the triangulation node x_i. An explicit approximation of (8) is

$$\omega_i^j|T'_i| - \omega_{i*}^{j-1}|T'_{i*}| = \delta t\,\nu L_s\omega_i^{j-1} + F_i. \tag{9}$$

In this, $F_i := \int_{T'_i} \tilde{f}\,\mathrm{d}x\,\mathrm{d}y$, $\omega_i^j := \omega(x_i, t_j)$ and $\omega_{i*}^{j-1} := \omega(x_i - \mathbf{u}_i^{j-1}\,\delta t, t_{j-1})$. The last quantity is computed by interpolation among the discrete ω^{j-1}-values. $|T'_{i*}|$ denotes the area of $|T'|$ at t_{j-1}. To first order, this area is polygonal. The vertices of this polygon are computed as $x'_k - u_k^{j-1}\,\delta t$, by analogy with the node x_i, where x'_k is a circumcentre. The velocity of the circumcentre and also of the node must be obtained by another interpolation.

The velocity interpolation can be easily done on the following basis. Suppose we consider a single triangle and three normal velocity components u_1, u_2, u_3. It is reasonable to ask when these are the components of a constant vector field inside the triangle. Since only two components fix the vector, the problem is overdetermined. However, it turns out, and is straightforward to check, that the three components belong to a constant field iff they have zero net flux, i.e. iff (1) vanishes. The field is unique in this case. Since this does hold in our computation, this is a convenient solution to the interpolation problem and we will use it.

For the ω-interpolation we can use piecewise linear interpolation on the triangles. The interpolation error will then be second-order and so of higher order than the other errors already committed.

Since the flow is incompressible, it might seem that the area $|T'_i|$ is constant and that the area calculation could be avoided in (9). In the discrete flow only the triangle areas remain constant. The co-volumes areas could vary within the limits set by the truncation error. It seems likely that a constant area assumption would therefore be within the truncation error. At the same time, the (potentially) variable area method should have better global conservation properties. Nevertheless, it can be a relatively costly calculation and one might wish to avoid it. Practical data on this point are lacking.

The remaining item is the vorticity boundary condition computation which is needed for the right-hand side of (9). This is also important for discretizing the vector Laplacian and is treated separately in the next section.

5. VORTICITY BOUNDARY CONDITIONS

To generate the vorticity boundary conditions, let V be a small domain containing the point x_0. Then we have

$$\omega(x_0)|V| \approx \int_{\partial V} \mathbf{u}\cdot\mathbf{t}\,ds. \tag{10}$$

This approximation is most accurate when x_0 is at the centroid of V, but is first-order accurate for any position of x_0 in V. We can use this arbitrariness to define the boundary values for ω.

In Figure 5 we identify the polygon $T' \equiv$ OABCDEFO with V, and x_0 with the point O. In general these *boundary co-volumes* may be non-convex. To approximate the line integral in (13), assume $\mathbf{u}|_\Gamma = \mathbf{g}$ and define

$$\omega_0 = (|T'|)^{-1}(g_A^{(t)}h_{OA} - g_A^{(n)}h'_{AB} + u_{BC}h'_{BC} + u_{CD}h'_{CD} + u_{DE}h'_{DE} + g_F^{(n)}h'_{EF} + g_F^{(t)}h_{FO}). \tag{11}$$

This time, $g^{(t)}$ and $g^{(n)}$ denote anticlockwise tangential and outward normal components of $\mathbf{g}$, and ω_0 is the approximate boundary value at O. In practice, $g^{(t)}$ and $g^{(n)}$ should be computed as averages on the mesh edges. A similar formulation is used at each boundary point of the mesh. In this way the tangential boundary condition on the velocity is used to compute the vorticity on the boundary. This is the only place where the tangential velocity appears. Naturally, this boundary condition is associated with the vorticity transport equation rather than the velocity equations.

A preliminary check on the accuracy of this boundary treatment has been made. We solved for the steady solution of

$$\omega_t + \mathbf{u}\cdot\nabla\omega = \Delta\omega + \tilde{f},$$
$$\operatorname{div}\mathbf{u} = 0,$$
$$\operatorname{curl}\mathbf{u} = \omega,$$
$$\mathbf{u}|_{\partial\Omega} = \mathbf{g},$$

where $\tilde{f}(x) = 4\sin x \sin y$. Ω, $\mathbf{g}$ and a typical mesh are illustrated in Figure 6. The exact solution is

$$u(x, y) = \sin x \cos y,$$
$$v(x, y) = -\cos x \sin y,$$
$$\omega(x, y) = 2\sin x \sin y.$$

The discrete equations are (3), (4), (5), (9) and (11). These we solved with $h = \pi/n$ for $n = 4, 8, 16, 32$ and 64. To measure the boundary error we used

$$\varepsilon_\Gamma^2 = \sum_{\text{boundary nodes}} (\omega - \omega^h)^2.$$

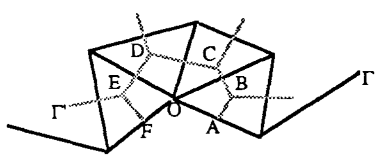

Figure 5

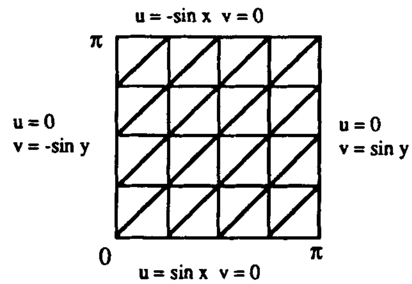

Figure 6

The results were as follows:

n	4	8	16	32	64
ε_Γ	0·038	0·113	0·0806	0·0462	0·0249
ρ		2·9	0·71	0·57	0·54

The last row gives the ratios of successive errors. The convergence may be asymptotically first-order in h. This can be compared with the $\sqrt{h}$ boundary accuracy for ω which is sometimes seen with other methods.

6. VECTOR LAPLACIAN

The vector Laplacian must be discretized for solving the primitive variable flow equations. We will discretize a particular component of it. Recall the definition

$$\Delta \mathbf{u} := \text{grad div}\,\mathbf{u} - \text{curl curl}\,\mathbf{u}.$$

In two dimensions, taking the inner product of this with a constant vector $\mathbf{n}$ gives

$$\mathbf{n}\cdot\Delta\mathbf{u} := \frac{\partial(\text{div}\,\mathbf{u})}{\partial n} - \frac{\partial(\text{curl}\,\mathbf{u})}{\partial t}, \tag{12}$$

where $\partial/\partial n$ denotes differentiation in the $\mathbf{n}$-direction and $\partial/\partial t$ denotes differentiation 90° to the left of it. Now consider Figure 7, which shows two triangles from the mesh. By the method of Section 3 we can compute the approximate divergence for each triangle and suppose it to hold at the circumcentres P and Q. Also, we can compute the approximate curls for the co-volumes and

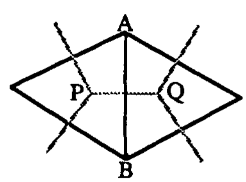

Figure 7

suppose them to hold at A and B. Next, choosing $\mathbf{n}$ to lie along PQ we can approximate (12) by

$$\frac{(\operatorname{div}\mathbf{u})_Q - (\operatorname{div}\mathbf{u})_P}{|PQ|} = \frac{(\operatorname{curl}\mathbf{u})_A - (\operatorname{curl}\mathbf{u})_B}{|AB|}. \tag{13}$$

Notice that in the second term we have differenced the vorticity. Before this term can be computed on edges with a boundary node, we have to know ω at the boundary nodes. The method of the previous section is used for this computation. This definition ensures that there is going to be a close connection between the primitive variable scheme and the vorticity–velocity scheme. We consider this in the next section.

7. CONNECTIONS

We will consider only the Stokes equations. This is necessary because our treatment of the convection term is special to the vorticity transport equation. First we will recall the co-volume discretization of the primitive equations from Reference 5. The Stokes equations are (with $\nu = 1$)

$$\mathbf{u}_t + \operatorname{grad} p = \Delta\mathbf{u} + \mathbf{f} \quad \text{in}\,\Omega, \tag{14}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in}\,\Omega, \tag{15}$$

$$\mathbf{u}|_\Gamma = \mathbf{g}. \tag{16}$$

The initial condition is $\mathbf{u}(\cdot, 0) = \mathbf{u}_0(\cdot)$. For the discretization, u denotes the vector of normal components, one for each triangle edge, and p denotes the vector of approximate pressures, one for each triangle. We take these to be defined at the circumcentres of the triangles. Denote by $\hat{C}_b$ the discrete curl operator as defined in Section 3 extended to compute boundary curls as defined in Section 5. Remembering that T differences along triangle edges, that S differences along co-edges and that F computes fluxes from u, we now have the following discretization of the normal components of (14)–(16):

$$\begin{aligned} u_t^{j-1} + Sp^{j-1} &= (S\hat{F} - T\hat{C}_b)u^{j-1} + f, \\ Fu^j &= 0, \\ u^j|_\Gamma &= g, \end{aligned} \tag{17}$$

where (13) has been used. Here f_i is defined to be the mean of the component of $\mathbf{f}$ along the co-edge labelled i, and g_j to be the mean of the component of $\mathbf{g}$ perpendicular to the boundary edge labelled j; u_t^{j-1} denotes a forward time difference. The mean normal components of the initial condition define u^0 in the natural way. We will assume that $\operatorname{div}\mathbf{u}_0 = 0$. Then $Fu^0 = 0$ and the first term on the right of (17) is zero for $j = 1, 2, \ldots$.

Now we will rewrite these equations in terms of a discrete vorticity $\omega := \hat{C}u$. Multiplying (17) by $\hat{C}$ and recalling that $\hat{C}S = 0$ we get

$$\omega_t = \hat{C}T\omega + \hat{C}f.$$

But $CT = L_s$ and

$$(Cf)_i = \int_{\partial T_i'} \mathbf{f}\cdot\mathbf{t}\,\mathrm{d}s = \int_{T_i'} \operatorname{curl}\mathbf{f}\,\mathrm{d}x\,\mathrm{d}y = F_i.$$

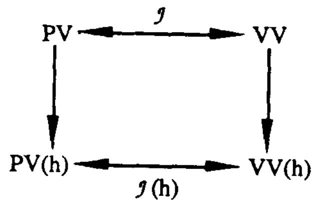

Figure 8

In view of these, we can now write the Stokes problem in the following *equivalent* form:

$$\omega_t^{j-1} = \hat{C} T \omega^{j-1} + F,$$
$$\omega^{j-1}|_\Gamma = \hat{C}_b u^{j-1}|_\Gamma,$$
$$F u^j = 0,$$
$$C u^j = \bar{\omega}^j,$$
$$u^j|_\Gamma = g.$$

The initial condition is $\omega(\cdot, 0) = \hat{C} u^0$. These equations are exactly those of the direct velocity–vorticity discretization of Sections 3–6, but they have been derived directly from the discrete Stokes equations instead of from the continuous velocity–vorticity equations. We can summarize these transformations in a diagram as illustrated in Figure 8. The vertical arrows denote co-volume discretization paths. The h-qualifier denotes a discretized entity. The transformation between the primitive variable and vorticity–velocity forms is represented by $\mathcal{T}$ and its discretization by $\mathcal{T}(h)$. This means that in the co-volume approach there is no preference for either of the two formulations of the flow equations. Solving one set of equations automatically solves the other. Whatever advantages (or disadvantages) are perceived for the discrete vorticity–velocity system are held by the discrete primitive variables system too and vice versa.

8. CONCLUSIONS

We have presented the basic discretization techniques of the co-volume approach. For isotropic problems this approach is characterized by its consistent use of single components of fields and equations instead of entire vectors and vector equations. It is applicable to vector field problems and potential equations derived from them. It is a co-ordinate-free method in the sense that it is the field operations themselves and not particular derivatives which are discretized. Basically, the co-volume method is a control volume method which makes use of more than one control volume. Theoretical convergence proofs have been obtained for several important kinds of linear problems. In this paper we have presented a fairly representative sample of algorithmic properties and manipulations which can be carried out in the co-volume framework.

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